

Article

Studying Chaos in a Modified Dynamical Model: Some Applications

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Abstract

Many authors consider modified SIQR (susceptible, infected, isolated (quarantined), and recovered individuals) models for childhood diseases. In this paper, we examine a modified differential system with N free parameters that may be of interest to epidemiology experts. We pay special attention to the Melnikov function, which corresponds to the proposed new model. We create the Melnikov equation ($M(t) = 0$) and analyze all of its roots using a specially designed software program. This gives the researcher the chance to accurately comprehend and articulate the classical Melnikov criterion for the potential appearance of chaos in the dynamical system. Additionally, we present a few specific modules for examining the new model's dynamics. Additionally covered is a potential use of the Melnikov function that corresponds to the differential model under consideration, with particular potential in the modeling and synthesis of antenna diagrams. Last but not least, we consider the proposed generalization from a stochastic point of view.

Keywords: dynamical systems with specific potential; homoclinic orbits; Melnikov function; antenna diagrams

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1. Introduction

Many authors consider modified SIQR models for childhood diseases.

The studies [1,2] contain comprehensive information for the reader, as well as additional bibliography.

In [2], the authors obtain an analytical understanding of the intricate dynamics that could result from the SIQR (susceptible, infected, isolated (quarantined), and recovered individuals) model.

Tens of thousands of publications on the topic of COVID-19 are known in the literature.

Analysis of the COVID-19 epidemic in different countries using the SIQR model has been the subject of research by many authors (see, for instance, refs. [3,4], where the reader can find additional relevant studies).

Homoclinic bifurcation exists in a biologically plausible region of the parameter and state space, as demonstrated using dynamical systems theory tools such as the center manifold reduction, normal form, and unfolding approaches.

We will not dwell on these classic techniques here.

For the global theory of Hopf bifurcation for differential equations, see, e.g., [5–15].

Melnikov analysis of chaos in a general epidemiological model can be found in [16–18].

More precisely, in [2] the authors consider the following differential model corresponding to the SIQR model:

$$\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = -y + x^2 - \epsilon(\alpha y + xy) \end{cases} \quad (1)$$

with Hamiltonian ($\epsilon = 0$)

$$H(x, y) = \frac{y^2}{2} + xy - \frac{x^3}{3},$$

where $0 \leq \epsilon < 1$ and $\alpha > 0$, and they studied the homoclinic bifurcation.

The separatrix cycle $\Gamma_0 \cap \{0\}$ corresponds to $H(x, y) = 0$, that is, motions on the curves defined by [2]

$$y^{\pm} = -x \pm \frac{\sqrt{3}}{3}x\sqrt{3+2x}; \quad -\frac{3}{2} \leq x \leq 0.$$

Research in this interesting direction can be successfully expanded.

In this article, we explore a modified differential system with N free parameters, which may be of interest to epidemiologists.

We pay special attention to the Melnikov function [19] corresponding to the proposed new model.

Chaos develops in the differential model if $M(t_0) = 0$ and $\frac{dM(t_0)}{dt_0} \neq 0$ for certain t_0 and parameter sets.

We use a specifically built software package to produce the Melnikov equation $M(t) = 0$ and investigate all of its roots.

This provides a chance for the researcher to accurately grasp and develop the classical Melnikov criterion for the occurrence of chaos in the dynamic system.

We also show some specialized modules for studying the dynamics of the new model.

A potential application of the Melnikov function to the examined differential model with a specified potential in antenna diagram modeling and synthesis is also discussed.

Furthermore, using the Euler presentation of the cos function and considering the weights used as the measure in a probability space, we transform the perturbation into a term that depends on the characteristic function of a discrete random variable. Alternatively, we might broaden the model by incorporating an arbitrary distribution. As a particular example of how this scheme works, we use the uniform distribution. In addition, we examine the model after adding a Gaussian perturbation in the second dynamics.

In this article, we offer an efficient strategy for training our master's and doctoral students on this interesting topic, and at the same time we hope to provoke a discussion on teaching methodology in the pages of the authoritative journal Axioms.

2. A New Model: Melnikov Approach, Some Simulations, and Challenges for Learners

We investigate the following new modified dynamical system in the following form:

$$\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = -y + x^2 - \epsilon \left(\alpha y + xy + y \sum_{j=1}^N a_j \cos(j\omega t) \right), \end{cases} \quad (2)$$

where $0 \leq \epsilon < 1$, $\alpha > 0$, $\omega > 0$, $a_i \geq 0$, $i = 1, 2, \dots, N$ and N is integer.

We obtain the homoclinic orbit through the following theorem:

Theorem 1. *The unperturbed form of System (2) has the following homoclinic orbit:*

$$\begin{aligned} x_h(t) &= -\frac{3}{2} \left(\operatorname{sech} \left(\frac{t}{2} \right) \right)^2 \\ y_h(t) &= \frac{3}{2} \left(\operatorname{sech} \left(\frac{t}{2} \right) \right)^2 \left(\tanh \left(\frac{t}{2} \right) + 1 \right). \end{aligned} \quad (3)$$

The saddle point is $(0, 0)$, and the center of the orbit is $(-1, 1)$.

Proof. The requirements $\dot{x} = 0$ and $\dot{y} = 0$ are only satisfied by the points $(0, 0)$ and $(-1, 1)$. The Jacobian of the unperturbed system is

$$J = \begin{pmatrix} 1 & 1 \\ 2x & -1 \end{pmatrix}. \quad (4)$$

The point $(0, 0)$ is the saddle point since it leads to a negative value for the determinant of Jacobian (4). Also, the point $(-1, 1)$ is the center of the homoclinic orbit because the determinant is positive.

To parametrize the orbit w.r.t. t , we need the auxiliary function $z = x + y$. Thus the unperturbed system, System (2), can be modified to

$$\begin{aligned} \dot{x} &= z \\ \dot{z} &= x^2 + x. \end{aligned} \quad (5)$$

Hence, $\ddot{x} = x^2 + x$. We shall use the classical approach of order reducing via the function z . We can write it as follows:

$$\ddot{x} = \frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} = z \frac{dz}{dx}. \quad (6)$$

Therefore,

$$z \frac{dz}{dx} = x^2 + x \quad (7)$$

or equivalently

$$z dz = (x^2 + x) dx. \quad (8)$$

Integrating in (8), we derive

$$z^2 = \frac{2}{3} x^3 + x^2 + C. \quad (9)$$

We can chose without any restrictions $C = 0$, and thus,

$$\dot{x} = \pm x \sqrt{\frac{2}{3} x + 1}. \quad (10)$$

Note that we have to impose the condition $x \geq -\frac{3}{2}$. We can further modify Equation (10) as

$$\int \frac{dx}{x\sqrt{\frac{2}{3}x+1}} = \pm \int dt. \quad (11)$$

For the first integral, we use the change in variables:

$$v = \sqrt{\frac{2}{3}x+1} \Leftrightarrow x = \frac{3}{2}(v^2-1). \quad (12)$$

Thus we derive

$$\int \frac{dx}{x\sqrt{\frac{2}{3}x+1}} = 2 \int \frac{dv}{v^2-1} = \ln \left| \frac{v-1}{v+1} \right| = \ln \left| \frac{\sqrt{2x+3}-\sqrt{3}}{\sqrt{2x+3}+\sqrt{3}} \right|. \quad (13)$$

Combining Formulas (11) and (13), we derive

$$\left| \frac{\sqrt{2x+3}-\sqrt{3}}{\sqrt{2x+3}+\sqrt{3}} \right| = e^{\pm t} \quad (14)$$

where accuracy to a constant can be ignored for our purposes. There are two functions that satisfy Equation (14):

$$\begin{aligned} x_1(t) &= -\frac{3}{2} \left(\operatorname{sech} \left(\frac{\pm t}{2} \right) \right)^2 \\ x_2(t) &= \frac{3}{2} \left(\operatorname{csch} \left(\frac{\pm t}{2} \right) \right)^2. \end{aligned} \quad (15)$$

The sign $\pm$ in both functions above vanishes since $\operatorname{sech}(\cdot)$ is an even function, $\operatorname{csch}(\cdot)$ is an odd one, and both of them are squared. In fact, the sign $\pm$ determines the direction of the movement along the orbit, but it does not change form. We conclude that the x part of the homoclinic orbit is given by $x_1(t)$ because the center of the orbit is the point $(-1, 1)$. Note that the condition $x \geq -\frac{3}{2}$ is satisfied. The z part is obtained by differentiating x , whereas y is derived through $y = z - x$. $\square$

Remark 1. The function $x_2(\cdot)$ leads to the outer branches, while Y leads to the unbounded part of the separatrix. Their dynamics can be obtained as

$$\begin{aligned} x(t) &= \frac{3}{2} \left(\operatorname{csch} \left(\frac{t}{2} \right) \right)^2 \\ y(t) &= -\frac{3}{2} \left(\operatorname{csch} \left(\frac{t}{2} \right) \right)^2 \left[\coth \left(\frac{t}{2} \right) + 1 \right]. \end{aligned} \quad (16)$$

Note that the functions $\operatorname{csch}(\cdot)$ and $\coth(\cdot)$ have singularities in the zero region.

The homoclinic orbit is shown in Figure 1.

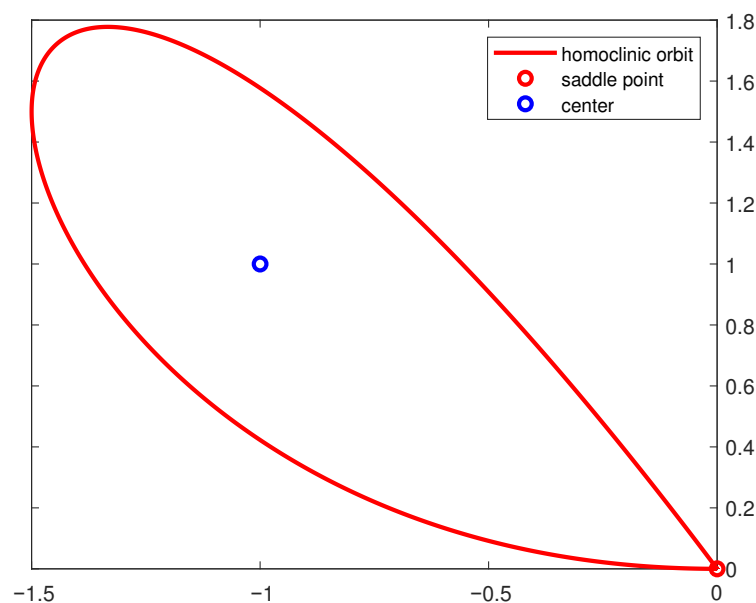


Figure 1. The homoclinic orbit.

2.1. Melnikov Approach

System (2) is of the form

$$\begin{cases} \frac{dx}{dt} = f_1(x, y) \\ \frac{dy}{dt} = f_2(x, y) - \epsilon g_2(x, y, t). \end{cases} \quad (17)$$

The Melnikov function [19] for Model (3) has the following form:

$$M(t_0) = \int_{-\infty}^{\infty} (x_h + y_h) \left(\alpha y_h + x_h y_h + y_h \sum_{j=1}^N a_j \cos(j\omega(t + t_0)) \right) dt, \quad (18)$$

where the homoclinic orbit $\{x_h, y_h\}$ is given by (3).

Remarks

1. For any arbitrary N , the reader can formulate the Melnikov criterion for the presence of chaos in the dynamical system under investigation.

2. Verifying the requirements $M(t_0) = 0$ and $\frac{dM(t_0)}{dt_0} \neq 0$ requires knowledge of all roots of specific nonlinear equations.

To identify all roots (simple or multiple) of nonlinear equations in confidence intervals, sophisticated numerical procedures are needed (see [20]).

3. Our planned WEB-based application [21] includes a customized module in CAS Mathematica.

The following is valid:

Proposition 1. If $N = 1$ and $\alpha > 0$, the solutions of the equation in Figure 2 represent the roots of the Melnikov function $M(t_0)$.

Example 1. Figure 3 depicts the Melnikov function $M(t_0)$ with $N = 1$, $\alpha = 0.9$, $\omega = 0.1$, and $a_1 = 3.1$.

We see that the roots (in confidence intervals) are simple.

The unstable and stable manifolds intersect transversally, and thus a differential system exhibits a chaotic behavior in the sense of the Smale–Birkhoff theorem [22–24] for small enough values of ϵ .

$$M(t_0) := \frac{3}{140} e^{-\frac{3}{2} i \omega} \left(28 a_1 - 48 e^{\frac{3}{2} i \omega} + 28 a_1 e^{\frac{3}{2} i \omega} + 56 e^{\frac{3}{2} i \omega} a - 70 i a_1 \omega + 70 i a_1 e^{\frac{3}{2} i \omega} \omega - 56 a_1 \omega^2 - 56 a_1 e^{\frac{3}{2} i \omega} \omega^2 + 70 i a_1 \omega^3 - 70 i a_1 e^{\frac{3}{2} i \omega} \omega^3 \right. \\ \left. + 28 a_1 \omega^4 + 28 a_1 e^{\frac{3}{2} i \omega} \omega^4 + 7 a_1 \omega (1 + \omega^2) (2 i + 5 \omega - 2 i \omega^2 + e^{\frac{3}{2} i \omega} (2 i - 5 \omega - 2 i \omega^2)) \text{PolyGamma}\left[0, 1 - \frac{i \omega}{2}\right] + 7 a_1 \omega (1 + \omega^2) (-2 i - 5 \omega + 2 i \omega^2 \right. \\ \left. + e^{\frac{3}{2} i \omega} (-2 i + 5 \omega + 2 i \omega^2)) \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i \omega}{2}\right] - 14 i a_1 \omega \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] - 14 i a_1 e^{\frac{3}{2} i \omega} \omega \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] - \right. \\ \left. 35 a_1 \omega^2 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] + 35 a_1 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] - 35 a_1 \omega^4 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] + 35 a_1 e^{\frac{3}{2} i \omega} \omega^4 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] \right. \\ \left. + 14 i a_1 \omega^3 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] + 14 i a_1 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] + 14 i a_1 \omega \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] + 14 i a_1 e^{\frac{3}{2} i \omega} \omega \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] \right. \\ \left. + 35 a_1 \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] - 35 a_1 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] + 35 a_1 \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] - 35 a_1 e^{\frac{3}{2} i \omega} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] - \right. \\ \left. 14 i a_1 \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] - 14 i a_1 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] \right) = 0$$

Figure 2. The answer to $M(t_0) = 0$ utilizing Proposition 1 of CAS Mathematica.

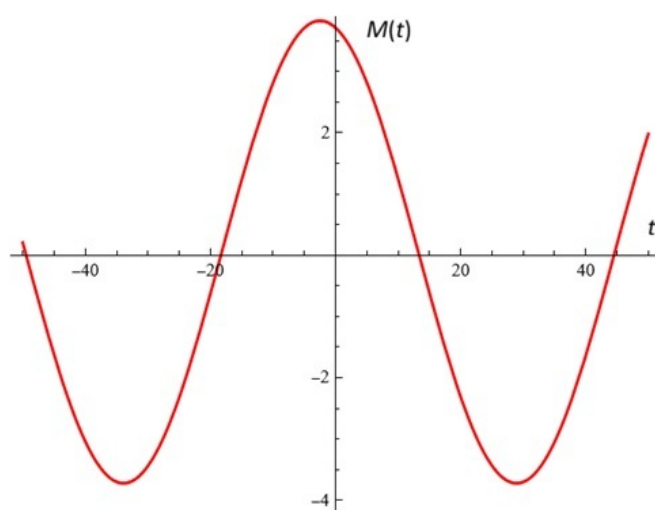


Figure 3. The Melnikov function $M(t_0)$ (Example 1).

The following is valid:

Proposition 2. If $N = 2$ and $\alpha > 0$, the solutions of the equation in Figure 4 represent the roots of the Melnikov function $M(t_0)$.

Example 2. Figure 5 shows the Melnikov function $M(t_0)$ for $N = 2$, $\alpha = 0.4$, $\omega = 0.75$, $a_1 = 3.1$, and $a_2 = 3.9$.

We see that $M(t_0)$ has single and multiple roots (in confidence intervals) of multiplicity two, and thus we have tangent contact between the stable and unstable manifolds.

This is an indicator for a complex (bifurcational) behavior.

$$M(t_0) := \frac{3}{140} e^{-\frac{3}{2} i \omega} \left(28 a_2 + 28 a_1 e^{\frac{3}{2} i \omega} - 48 e^{\frac{3}{2} i \omega} + 28 a_1 e^{\frac{3}{2} i \omega} + 56 e^{\frac{3}{2} i \omega} a - 140 i a_2 \omega + 70 i a_1 e^{\frac{3}{2} i \omega} \omega + 140 i a_2 e^{\frac{3}{2} i \omega} \omega - 224 a_2 \omega^2 - 56 a_1 e^{\frac{3}{2} i \omega} \omega^2 - 56 a_1 e^{\frac{3}{2} i \omega} \omega^3 \right. \\ \left. - 224 a_2 e^{\frac{3}{2} i \omega} \omega^3 + 560 i a_2 \omega^3 + 70 i a_1 e^{\frac{3}{2} i \omega} \omega^3 - 560 i a_2 e^{\frac{3}{2} i \omega} \omega^3 + 448 a_2 \omega^4 + 28 a_1 e^{\frac{3}{2} i \omega} \omega^4 + 28 a_1 e^{\frac{3}{2} i \omega} \omega^5 + 448 a_2 e^{\frac{3}{2} i \omega} \omega^5 + 7 a_1 e^{\frac{3}{2} i \omega} \omega (1 + \omega^2) \right. \\ \left. + e^{\frac{3}{2} i \omega} (2 i - 5 \omega - 2 i \omega^2) \text{PolyGamma}\left[0, 1 - \frac{i \omega}{2}\right] + 7 a_1 e^{\frac{3}{2} i \omega} \omega (1 + \omega^2) (-2 i - 5 \omega + 2 i \omega^2 + e^{\frac{3}{2} i \omega} (-2 i + 5 \omega + 2 i \omega^2)) \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i \omega}{2}\right] - 14 i a_1 \omega \text{PolyGamma}\left[0, 1 - \frac{i \omega}{2}\right] \right. \\ \left. - 14 i a_1 e^{\frac{3}{2} i \omega} \omega \text{PolyGamma}\left[0, 1 - \frac{i \omega}{2}\right] - 35 a_1 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, 1 - \frac{i \omega}{2}\right] + 35 a_1 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, 1 - \frac{i \omega}{2}\right] - 35 a_1 e^{\frac{3}{2} i \omega} \omega^4 \text{PolyGamma}\left[0, 1 - \frac{i \omega}{2}\right] + 35 a_1 e^{\frac{3}{2} i \omega} \omega^4 \text{PolyGamma}\left[0, 1 - \frac{i \omega}{2}\right] \right. \\ \left. + 14 i a_1 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, 1 - \frac{i \omega}{2}\right] + 14 i a_1 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, 1 - \frac{i \omega}{2}\right] + 14 i a_1 e^{\frac{3}{2} i \omega} \omega \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i \omega}{2}\right] + 14 i a_1 e^{\frac{3}{2} i \omega} \omega \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i \omega}{2}\right] + 35 a_1 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i \omega}{2}\right] \right. \\ \left. - 35 a_1 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i \omega}{2}\right] + 35 a_1 e^{\frac{3}{2} i \omega} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i \omega}{2}\right] - 35 a_1 e^{\frac{3}{2} i \omega} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i \omega}{2}\right] - 14 i a_1 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i \omega}{2}\right] - 14 i a_1 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i \omega}{2}\right] \right. \\ \left. + 28 i a_2 \omega \text{PolyGamma}\left[0, 1 - i \omega\right] + 28 i a_2 e^{\frac{3}{2} i \omega} \omega \text{PolyGamma}\left[0, 1 - i \omega\right] + 140 a_2 \omega^2 \text{PolyGamma}\left[0, 1 - i \omega\right] - 140 a_2 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, 1 - i \omega\right] + 560 a_2 \omega^3 \text{PolyGamma}\left[0, 1 - i \omega\right] - 560 a_2 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, 1 - i \omega\right] \right. \\ \left. - 448 i a_2 \omega^3 \text{PolyGamma}\left[0, 1 - i \omega\right] - 448 i a_2 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, 1 - i \omega\right] - 28 i a_2 \omega \text{PolyGamma}\left[0, \frac{3}{2} - i \omega\right] - 28 i a_2 e^{\frac{3}{2} i \omega} \omega \text{PolyGamma}\left[0, \frac{3}{2} - i \omega\right] - 140 a_2 \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} - i \omega\right] + 140 a_2 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} - i \omega\right] \right. \\ \left. - 560 a_2 \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} - i \omega\right] + 560 a_2 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} - i \omega\right] + 448 i a_2 \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} - i \omega\right] + 448 i a_2 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} - i \omega\right] - 28 i a_2 \omega \text{PolyGamma}\left[0, 1 + i \omega\right] - 28 i a_2 e^{\frac{3}{2} i \omega} \omega \text{PolyGamma}\left[0, 1 + i \omega\right] \right. \\ \left. - 140 a_2 \omega^2 \text{PolyGamma}\left[0, 1 + i \omega\right] + 140 a_2 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, 1 + i \omega\right] + 560 a_2 \omega^3 \text{PolyGamma}\left[0, 1 + i \omega\right] + 560 a_2 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, 1 + i \omega\right] + 448 i a_2 \omega^3 \text{PolyGamma}\left[0, 1 + i \omega\right] + 448 i a_2 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, 1 + i \omega\right] \right. \\ \left. + 28 i a_2 \omega \text{PolyGamma}\left[0, \frac{3}{2} + i \omega\right] + 28 i a_2 e^{\frac{3}{2} i \omega} \omega \text{PolyGamma}\left[0, \frac{3}{2} + i \omega\right] + 140 a_2 \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + i \omega\right] - 140 a_2 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + i \omega\right] + 560 a_2 \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} + i \omega\right] - 560 a_2 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} + i \omega\right] \right. \\ \left. - 448 i a_2 \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} + i \omega\right] - 448 i a_2 e^{\frac{3}{2} i \omega} \omega^3 \text{PolyGamma}\left[0, \frac{3}{2} + i \omega\right] \right) = 0$$

Figure 4. The answer to $M(t_0) = 0$ using Proposition 2 of CAS Mathematica.

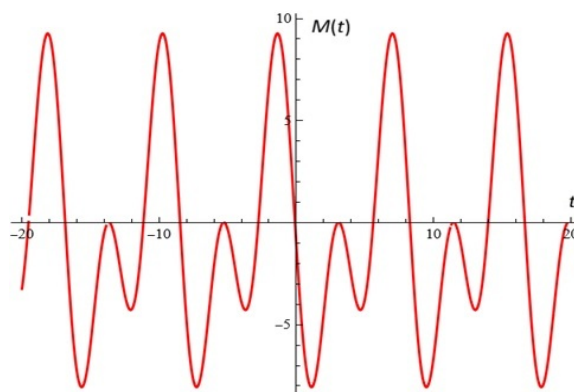


Figure 5. The Melnikov function $M(t_0)$ (Example 2).

The following is valid:

Proposition 3. The solutions of the equation in Figure 6 are the roots of the Melnikov function $M(t_0)$ if $N = 3$ and $\alpha > 0$.

Example 3. Figure 7 depicts the Melnikov function $M(t_0)$ for $N = 3$, $\alpha = 0.5$, $\omega = 0.54$, $a_1 = 3.9$, $a_2 = 0.9$, and $a_3 = 2.8$, respectively. We can observe that $M(t_0)$ has single and multiple roots of multiplicity two.

$$M(t_0) := \frac{3}{140} e^{-3 \frac{3 \sqrt{3}}{2} i t} \left(28 a_3 + 28 a_2 e^{\frac{3 \sqrt{3}}{2} i t} + 28 a_1 e^{\frac{3 \sqrt{3}}{2} i t} - 48 e^{\frac{3 \sqrt{3}}{2} i t} + 28 a_1 e^{\frac{3 \sqrt{3}}{2} i t} + 28 a_2 e^{\frac{3 \sqrt{3}}{2} i t} + 28 a_3 e^{\frac{3 \sqrt{3}}{2} i t} + 56 e^{\frac{3 \sqrt{3}}{2} i t} - 210 i a_3 u - 140 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u - 70 i a_1 e^{\frac{3 \sqrt{3}}{2} i t} u + 140 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 + 210 i a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 - 504 a_3 u^3 - 224 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 - 56 a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 - 224 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 - 504 a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 + 1990 i a_3 u^3 + 560 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 + 70 i a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 - 70 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 - 560 i a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 - 1990 i a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 + 2268 a_3 u^3 + 448 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 + 28 a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 + 448 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 + 2268 a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 + 7 a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \left(2 i - 5 u - 2 i u^2 + e^{\frac{3 \sqrt{3}}{2} i t} (2 i - 5 u - 2 i u^2) \right) \text{PolyGamma}\left[0, 1 - \frac{i u}{2}\right] + 7 a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \left(-2 i - 5 u + 2 i u^2 + e^{\frac{3 \sqrt{3}}{2} i t} (-2 i - 5 u + 2 i u^2) \right) \text{PolyGamma}\left[0, 1 + \frac{i u}{2}\right] - 14 i a_1 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, 1 - \frac{i u}{2}\right] - 14 i a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - \frac{i u}{2}\right] - 35 a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + \frac{i u}{2}\right] + 35 a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - \frac{i u}{2}\right] + 35 a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + \frac{i u}{2}\right] + 14 i a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - \frac{i u}{2}\right] + 14 i a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + \frac{i u}{2}\right] - 35 a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - \frac{i u}{2}\right] - 35 a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + \frac{i u}{2}\right] + 14 i a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - \frac{i u}{2}\right] - 14 i a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + \frac{i u}{2}\right] - 14 i a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - \frac{i u}{2}\right] - 14 i a_1 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + \frac{i u}{2}\right] + 28 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, 1 - i u\right] + 28 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, 1 - i u\right] + 140 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - i u\right] - 140 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - i u\right] + 560 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - i u\right] - 560 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - i u\right] - 448 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - i u\right] - 448 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - i u\right] - 28 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, \frac{3}{2} - i u\right] - 28 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, \frac{3}{2} - i u\right] - 140 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} - i u\right] - 140 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} - i u\right] + 560 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} - i u\right] + 560 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} - i u\right] - 28 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, 1 + i u\right] - 28 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, 1 + i u\right] - 140 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + i u\right] - 140 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + i u\right] + 560 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + i u\right] + 560 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + i u\right] - 448 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + i u\right] - 448 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 + i u\right] + 28 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, \frac{3}{2} + i u\right] + 28 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, \frac{3}{2} + i u\right] + 140 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} + i u\right] + 140 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} + i u\right] - 560 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} + i u\right] - 560 a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} + i u\right] - 448 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} + i u\right] - 448 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} + i u\right] + 28 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, 1 - \frac{3 \sqrt{3}}{2}\right] + 28 i a_2 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, 1 - \frac{3 \sqrt{3}}{2}\right] + 315 a_3 u^3 \text{PolyGamma}\left[0, 1 - \frac{3 \sqrt{3}}{2}\right] - 315 a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - \frac{3 \sqrt{3}}{2}\right] + 2835 a_3 u^3 \text{PolyGamma}\left[0, 1 - \frac{3 \sqrt{3}}{2}\right] - 2835 a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - \frac{3 \sqrt{3}}{2}\right] - 3402 i a_3 u^3 \text{PolyGamma}\left[0, 1 - \frac{3 \sqrt{3}}{2}\right] - 3402 i a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, 1 - \frac{3 \sqrt{3}}{2}\right] - 42 i a_3 u \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3 \sqrt{3}}{2}\right] - 42 i a_3 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3 \sqrt{3}}{2}\right] - 42 i a_3 u \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3 \sqrt{3}}{2}\right] - 42 i a_3 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3 \sqrt{3}}{2}\right] + 315 a_3 u^3 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3 \sqrt{3}}{2}\right] + 315 a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3 \sqrt{3}}{2}\right] + 2835 a_3 u^3 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3 \sqrt{3}}{2}\right] + 2835 a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3 \sqrt{3}}{2}\right] - 3402 i a_3 u^3 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3 \sqrt{3}}{2}\right] - 3402 i a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3 \sqrt{3}}{2}\right] - 42 i a_3 u \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] - 42 i a_3 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] - 42 i a_3 u \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] - 42 i a_3 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] + 315 a_3 u^3 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] + 315 a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] + 2835 a_3 u^3 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] + 2835 a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] - 3402 i a_3 u^3 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] - 3402 i a_3 e^{\frac{3 \sqrt{3}}{2} i t} u^3 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] - 42 i a_3 u \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] - 42 i a_3 e^{\frac{3 \sqrt{3}}{2} i t} u \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3 \sqrt{3}}{2}\right] \right) = 0$$

Figure 6. The answer to $M(t_0) = 0$ using Proposition 3 of CAS Mathematica.

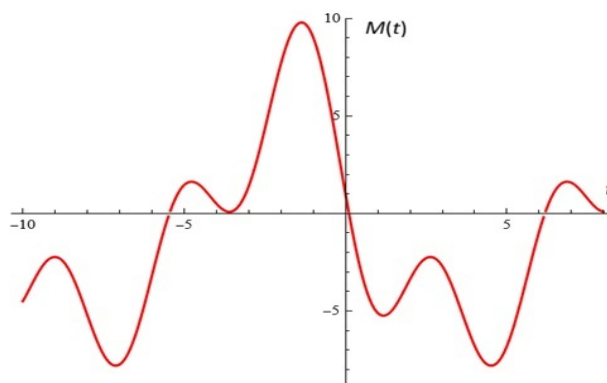


Figure 7. The Melnikov function $M(t_0)$ (Example 3).

Example 4. Figure 8 shows the Melnikov function $M(t_0)$ for $N = 2$, $\alpha = 1.4$, $\omega = 0.82$, $a = 10 * [2, 3]$, $\epsilon = 0.01$, $x_0 = -1.1797$, and $y_0 = 1.7248$.

Example 5. Figure 9 shows the Melnikov function $M(t_0)$ for $N = 3$, $\alpha = 0.3$, $\epsilon = 0.0$, $\omega = 0.9$, $a = 20 * [1.1, 0.9, 2]$, $x_0 = -1.1797$, and $y_0 = 1.7248$.

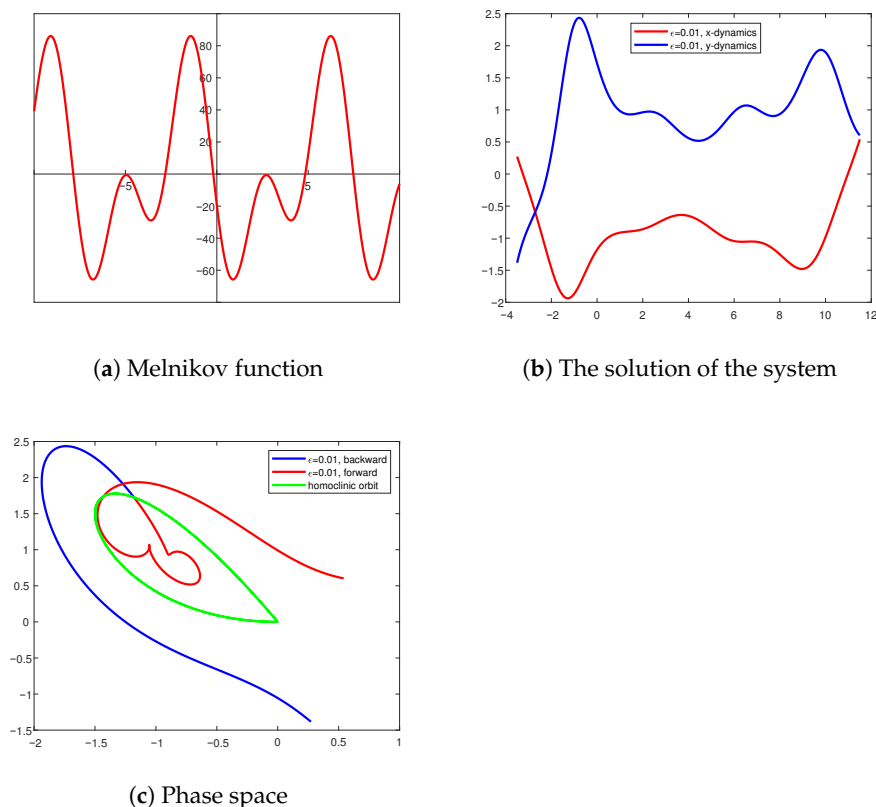


Figure 8. Example 4.

Remarks

1. From Propositions 1–3, we see that the Melnikov function is expressed by the *PolyGamma* function, where *PolyGamma* $[n, z]$ gives the n -th derivative of the *digamma* function $\psi^{(n)}(z)$, i.e., $\psi^{(0)}(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$.

For more details, see [25,26].

2. The explicit encoding of $M(t_0)$ (particularly for large values of the parameter N) is time-intensive.

In addition, the user needs to specify reasonable restrictions on some of the parameters of the studied differential model when calculating such integrals in the intended modules implemented in the existing software environments for scientific calculations.

One possibility in this direction is the restriction $|\text{Im}(\omega)| < \text{Const}$.

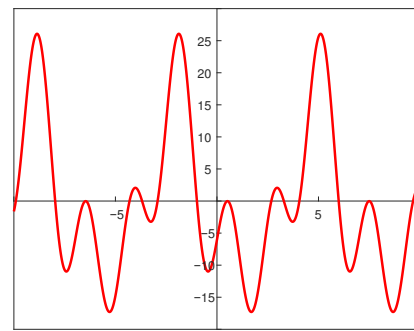
As we have already noted in our other articles, the user does not have to be a professional mathematician, which requires upgrading the existing modules.

Figures 2, 4 and 6 illustrate the calculation of $M(t_0)$ using CAS Mathematica.

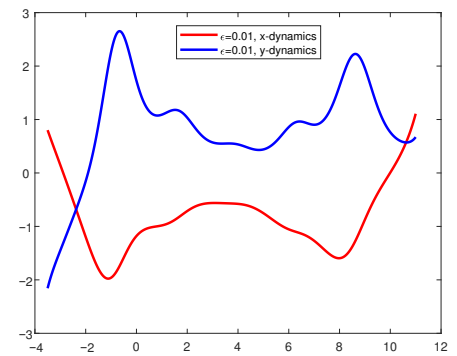
We include these figures to illustrate the difficulties that the user encounters.

The above-mentioned WEB application [21] was used significantly to generate the equation $M(t_0) = 0$ and to analyze its roots (simple or multiple).

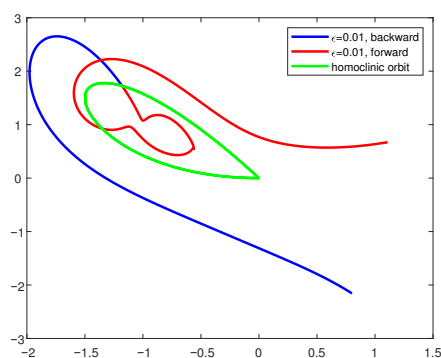
The modified SIQR model proposed by us is hypothetical. We expect specialists working in the field of epidemiology to express their opinion regarding its validity.



(a) Melnikov function



(b) The solution of the system



(c) Phase space

Figure 9. Example 5.

2.2. Some Simulations

Example 6. Figure 10 shows the system's simulations (Equation (2)) for $x_0 = 0.1$ and $y_0 = 0.1$ and for $N = 3$, $\alpha = 0.9$, $\epsilon = 0.1$, $\omega = 1.05$, $a_1 = 1.4$, $a_2 = 1.2$, and $a_3 = 1.1$.

Example 7. For $N = 5$, $\alpha = 0.8$, $\epsilon = 0.09$, $\omega = 1.02$, $a_1 = 2.4$, $a_2 = 0.2$, $a_3 = 3.1$, $a_4 = 0.1$, and $a_5 = 4.1$, the system's simulations (Equation (2)) for $x_0 = 0.1$ and $y_0 = 0.1$ are depicted in Figure 11.

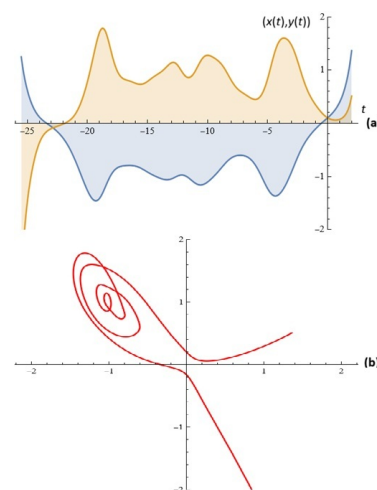


Figure 10. (a) solutions of System (2); (b) phase space (Example 6).

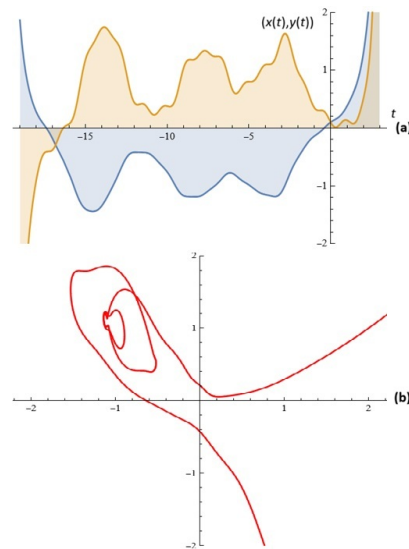


Figure 11. (a) solutions of System (2); (b) phase space (Example 7).

2.3. Challenges for Learners

We provide an effective learning technique that prioritizes learning while challenging our master's and doctoral students to address the triangle of enigmas, creativity, and acmeology.

After giving curricular modules, we will assign the following activities for self-learning.

Task 1. (self-learning): Examine the behavior of Model (2) at these values: $N = 7$; $\alpha = 0.75$; and $a_1 = 0.4$, $a_2 = 0.2$, $a_3 = 0.1$, $a_4 = 9.1$, $a_5 = 0.1$, $a_6 = 0.4$, and $a_7 = 0.1$:

(a) What values of the parameters ϵ and ω yield the phase portrait displayed in Figure 12b?

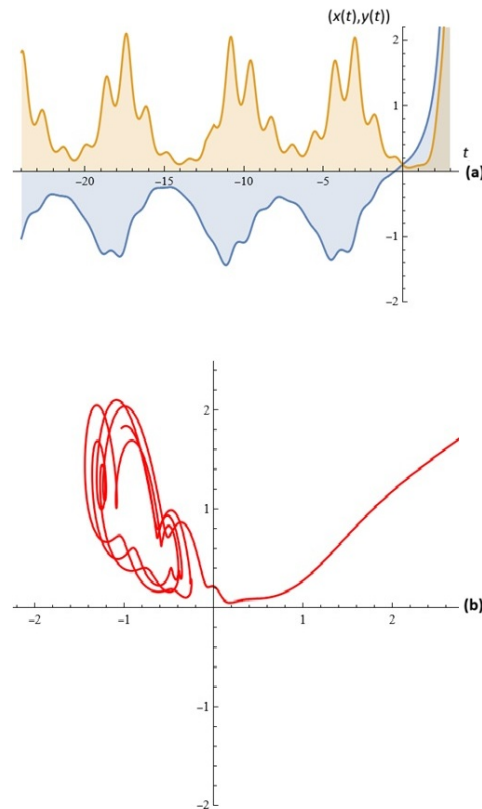


Figure 12. (a) x and y : components of the solution of System (2); (b) phase space (Task 1 (self-learning)).

(Answer: When you complete the exercise successfully, you should get the following approximate values: $\epsilon = 0.16$ and $\omega = 1.2$.)

(b) Demonstrate a chaotic phase portrait/sensitive reliance for parameters above this level;

(c) Draw appropriate conclusions.

As teachers, we conclude that the proposed methodology helps pupils adapt rather effectively and touches on Melnikov's elegant explanation for the possibility of chaos in the dynamical system under investigation.

3. Some Applications

The hypothetical normalized antenna factor can be defined as $M^*(\theta) = \frac{1}{D}|M(K \cos \theta + k_2)|$, where θ is the azimuth angle; $K = k_1 d$; $k_1 = \frac{2\pi}{\lambda}$; λ is the wave length; d is the distance between emitters; and k_2 represents the phase difference.

Over the past 2–3 years, in a number of our publications, we have been promoting the idea of a possible application of Melnikov polynomials for generating antenna factors [27–29].

The typical N -element linear phased array factor used to determine A_k coefficients is

$$AF(\theta) = \sum_{k=1}^{\frac{N}{2}} A_k \cos((2k-1)u) = M(x),$$

where $u = \frac{\pi d}{\lambda} \cos \theta$, and $x = x_0 \cos u$, where x_0 is the design parameter.

This concept was inspired by the classic Dolph–Chebyshev antenna array [30–32].

Due to the fact that each specific Melnikov polynomial corresponds to a specific differential model and that the number of classical and newer models with specific potential described in the literature is enormous, the variety of model functions is obviously also large.

We will explicitly note that in the article cited above, the reader can find information about all the numerical methods used in our planned WEB-based application from the field of approximation theory, that is, approximations with constraints and optimization theory.

Example 8. Let us fix $N = 5$, $\alpha = 0.99$, $\omega = 0.18$, $a_1 = 1$, $a_2 = 0.45$, $a_3 = 0.45$, $a_4 = 0.45$, $a_5 = 1.2$, $K = 9.1$, and $k_2 = -1.7$.

The Melnikov function $M(t_0)$ is shown in Figure 13a. The Melnikov antenna factor is displayed in Figure 13b.

Challenges for Learners

After providing curriculum modules, we will set the following tasks for self-learning.

Task 2 (self-learning with increased difficulty): Let $N = 7$:

(a) Generate the Melnikov function $M(t_0)$;

(b) Generate a Melnikov antenna factor with possible low levels on the side leaves (see, for example, Figure 14).

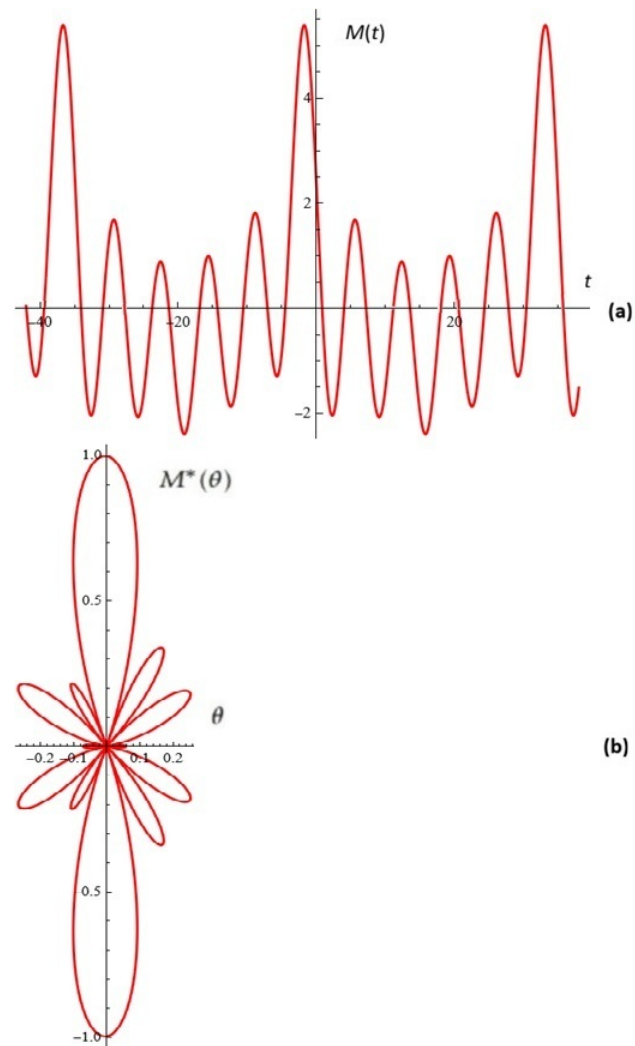


Figure 13. (a) Melnikov function $M(t_0)$ (Example 8); (b) Melnikov antenna factor (Example 8).

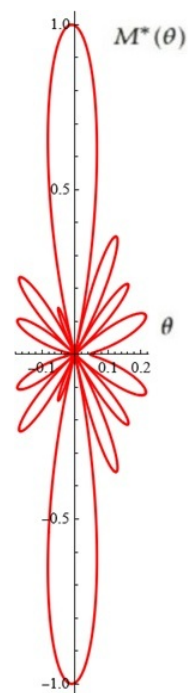


Figure 14. Melnikov antenna factor (Task 2).

4. A Generalized Model

We shall now present a further generalization of Model (2). Let $\bar{a} = \sum_{j=1}^N a_j$ and $b_j = \frac{a_j}{\bar{a}}$. Obviously, $\sum_{j=1}^N b_j = 1$, and thus, we can consider these constants as the probabilities that some random variables can achieve the value j . Note that the domain is the set $\{1, 2, \dots, N\}$. Using the Euler presentation of the cos function

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad (19)$$

we can transform the original model (Model (2)) into

$$\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= -y + x^2 - \epsilon \left(\alpha y + xy + \bar{a} y \sum_{j=1}^N a_j \cos(j\omega t) \right) \\ &= -y + x^2 - \epsilon \left(\alpha y + xy + \bar{a} y \frac{\Psi(\omega t) + \Psi(-\omega t)}{2} \right), \end{aligned} \quad (20)$$

where $\Psi(\cdot)$. We denote the characteristic function of the above-mentioned random variable. Presentation (20) allows us to define Model (2) for random variables stated on an arbitrary domain. In light of this, Melnikov's integral (Equation (18)) can be written as

$$M(t_0) = \int_{-\infty}^{\infty} (x_h + y_h) \left(\alpha y_h + x_h y_h + \bar{a} y_h \frac{\Psi(\omega(t+t_0)) + \Psi(-\omega(t+t_0))}{2} \right) dt, \quad (21)$$

where the homoclinic orbit is defined by Formula (3). To get more detailed results, we need to know the particular distribution. We shall give a simple example based on the continuous uniform distribution on the interval (c, d) . Its characteristic function is

$$\Psi(x) = \begin{cases} \frac{e^{idx} - e^{icx}}{x(d-c)}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases} \quad (22)$$

and thus,

$$\frac{\Psi(x) + \Psi(-x)}{2} = \begin{cases} \frac{\sin(dx) - \sin(cx)}{x(d-c)}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0. \end{cases} \quad (23)$$

This model is presented in Figures 15 and 16 for the following parameter values: $\alpha = 1.01$, $\omega = 1$, $\bar{a} = 10$, $c = -1$, $d = 2$, and $\epsilon = 0.01$.

The Melnikov function can be seen in Figure 15.

Different dynamics and the phase portraits w.r.t. to the initial condition are depicted in Figure 15.

For the first row, we have $\{x(0) = -1.1797; y(0) = 1.7248\}$ —the point is at the homoclinic orbit.

For the second row, $\{x(0) = -1.1897; y(0) = 1.6248\}$ —the point is in the interior of the orbit.

Finally, for the third row, we use an initial value outside the homoclinic orbit— $\{x(0) = -1.1797; y(0) = 1.7248\}$.

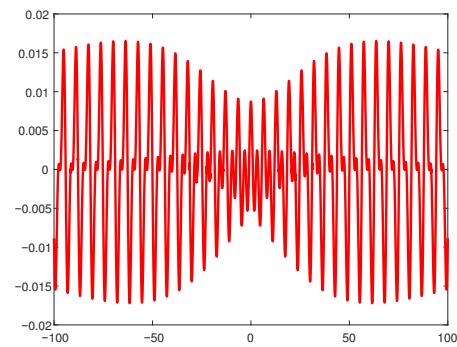
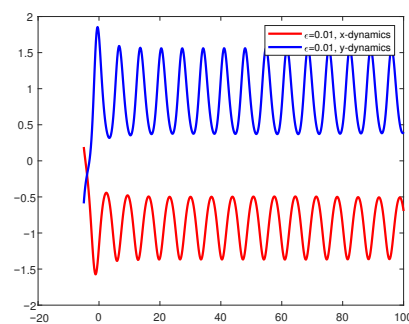
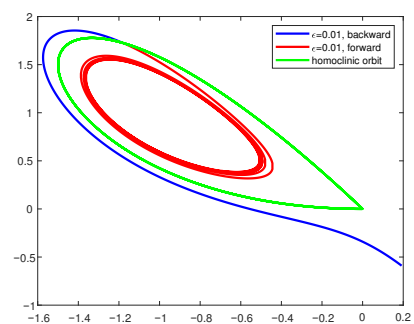


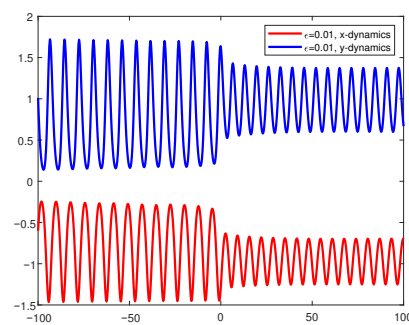
Figure 15. Melnikov function.



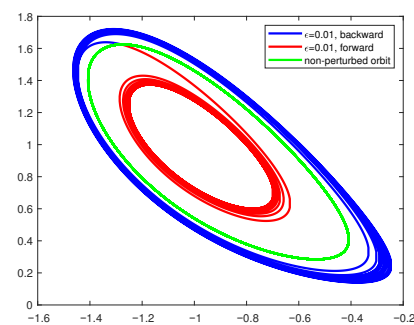
(a) Dynamics



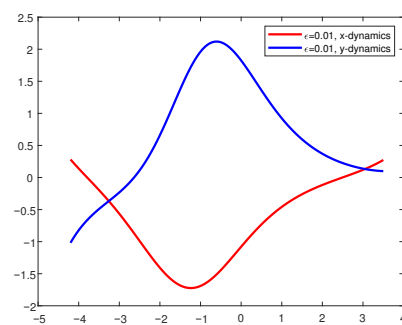
(b) Phase portrait



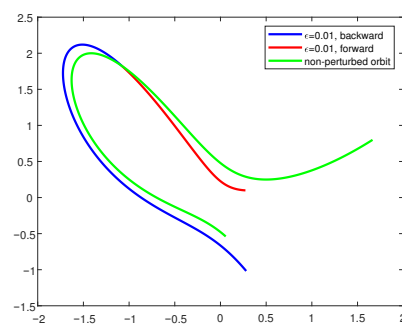
(c) Dynamics



(d) Phase portrait



(e) Dynamics



(f) Phase portrait

Figure 16. Dynamics and phase portraits.

Last but not least, let us introduce a simple perturbation based on the Gaussian distribution with parameters (μ, σ^2) in the x term. Its characteristic function is

$$\Psi(x) = e^{i\mu x - \frac{\sigma^2 x^2}{2}} \quad (24)$$

and therefore,

$$\frac{\Psi(x) + \Psi(-x)}{2} = e^{-\frac{\sigma^2 x^2}{2}} \cos(\mu x). \quad (25)$$

Thus, System (20) turns into

$$\begin{aligned} \frac{dx}{dt} &= x + y + \epsilon \bar{A} e^{-\frac{\sigma^2 v^2 t^2}{2}} \cos(\mu v t) \\ \frac{dy}{dt} &= -y + x^2 - \epsilon \left(\alpha y + x y + \bar{a} y \frac{\sin(d\omega t) - \sin(c\omega t)}{\omega t(d-c)} \right) \end{aligned} \quad (26)$$

under the consideration $\frac{\sin(d\omega t) - \sin(c\omega t)}{\omega t(d-c)} = 1$ when $t = 0$. The constant $\bar{A} \geq 0$ is the scaling factor for the Gaussian term, whereas ν is the frequency converter. The Melnikov integral of System (26) is

$$M(t_0) = \int_{-\infty}^{\infty} \left\{ \bar{A} (x_h^2 - y_h) e^{-\frac{\sigma^2 v^2 (t+t_0)^2}{2}} \cos(\mu v (t+t_0)) + (x_h + y_h) \left(\alpha y_h + x_h y_h + \bar{a} y_h \frac{\sin(d\omega(t+t_0)) - \sin(c\omega(t+t_0))}{\omega(t+t_0)(d-c)} \right) \right\} dt. \quad (27)$$

In Figure 17, we present Model (26): Figure 17a presents the dynamics, whereas Figure 17b is for the phase portrait. The initial point is chosen to be the inner part of the homoclinic orbit. The parameters chosen for the Gaussian perturbation are $\bar{A} = 10$, $\nu = 1$, $\mu = 1$, and $\sigma = 1$. All other parameters are preserved. Despite some similarities with the model without the x perturbation (Figure 16c,d), we observe that the Gaussian term has a significant impact on the dynamics. The Melnikov integral (Equation (27)) for this example is presented in Figure 17c. We can observe a rich structure that indicates a complicated chaotic behavior.

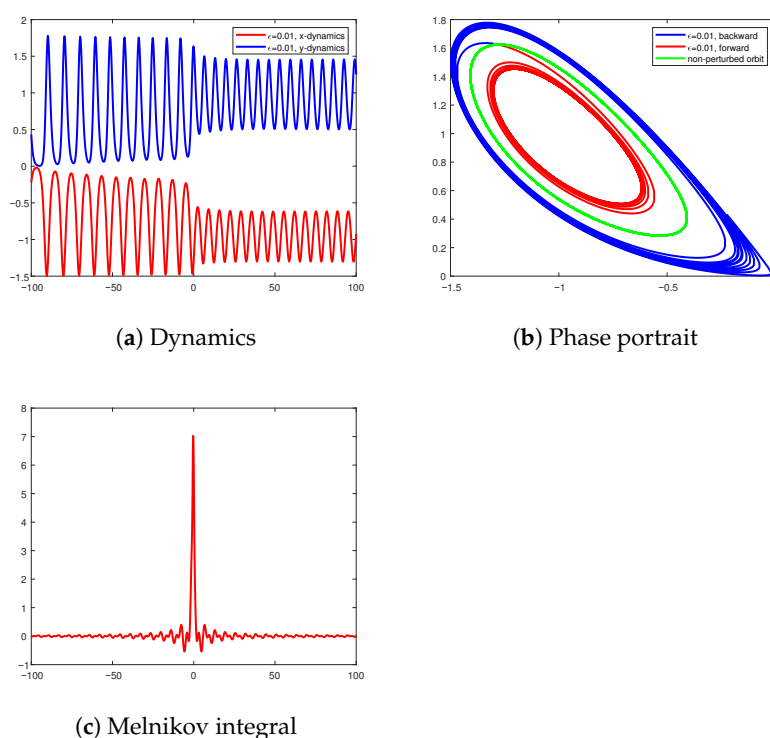


Figure 17. Dynamics, phase portrait, and the Melnikov function: double perturbation.

5. Conclusions

This paper investigates the prediction of chaos in a modified differential system with a specified potential and N free parameters, which may be of interest to epidemiologists.

We pay special emphasis to the Melnikov function, which corresponds to the suggested novel model.

Several simulations are combined. We also demonstrate several specific modules for studying the dynamics of the proposed model.

A possible application of the Melnikov function to the examined differential model with a specified potential in antenna diagram modeling and synthesis is also discussed.

Of course, it is still unclear whether such diagrams can be implemented in practice.

In future publications, we aim to explore additional differential systems with potential applications to modified SIR/SIRD/SEIR/GSEIR models and “growth models” [33–54].

We envisage future research on modified SIQR-type models [3,4] in light of the considerations in this article.

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